



All 1/2 BPS solutions of IIB supergravity with $SO(4) \times SO(4)$ isometry

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Received 8 October 2007; accepted 18 October 2007

Available online 26 October 2007

Editor: M. Cvetič

Abstract

In hep-th/0409174, Lin, Lunin and Maldacena constructed a set of regular 1/2 BPS geometries in IIB theory. These remarkable ‘bubbling AdS’ geometries have a natural interpretation as duals of chiral primary operators with weight $\Delta = J$ in $\mathcal{N} = 4$ super-Yang–Mills theory. Although these geometries have been assumed to be complete, from a purely supergravity point of view, additional 1/2 BPS configurations may potentially exist with a preferred null isometry. We explore this possibility and prove that the only additional class of 1/2 BPS solutions with $SO(4) \times SO(4)$ isometry are the familiar IIB pp-waves.

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1. Introduction

Motivated by the AdS/CFT correspondence, Lin, Lunin and Maldacena (LLM) [1] recently explored the space of 1/2 BPS supergravity solutions dual to chiral primary operators with weight $\Delta = J$ in $\mathcal{N} = 4$ super-Yang–Mills theory. These solutions may be viewed as 1/2 BPS excitations of $AdS_5 \times S^5$, and are described in terms of a ‘droplet’ picture. In particular, all such regular 1/2 BPS supergravity configurations are completely determined in terms of boundary data (droplets) specified on a two-dimensional (x_1-x_2) plane. As befits the AdS/CFT correspondence, this two-dimensional plane may be identified with the phase space of free fermions that arise in the dual description of the corresponding 1/2 BPS states in the $\mathcal{N} = 4$ gauge theory [2,3].

Although the 1/2 BPS geometries of IIB theory which are constructed in [1] form a complete set of duals to the $\Delta = J$ chiral primaries, they do not necessarily comprise *all* 1/2 BPS solutions of IIB supergravity. For example, the basic super-

gravity D3-brane itself is perfectly regular, and smoothly interpolates between $AdS_5 \times S^5$ and 10-dimensional Minkowski space. Of course, this full D3-brane geometry does not admit an $SO(4) \times SO(4)$ isometry that is relevant to $\Delta = J$ states, and hence does not fall in the class of ‘Bubbling AdS’ solutions considered in [1]. Nevertheless, it is still an open possibility whether there are any additional solutions with $SO(4) \times SO(4)$ isometry that have not yet been identified. (An interesting new class of ‘bubbling orientifold’ solutions were found in [4]; these solutions are locally equivalent to the bubbling AdS solutions, but have global identifications. Since we restrict ourselves to a local analysis, we do not see this distinction in the solutions below. Nevertheless, a full classification of geometries ought to take such global issues into account.)

The possibility that there are additional regular 1/2 BPS solutions of IIB supergravity arises from two observations. The first observation is that the analysis of [1] only takes into account the metric and self-dual five-form. In principle, one could expect new solutions to arise that involve the three-forms as well as the IIB dilaton-axion field. It was shown in [5], however, that excitations of the dilaton-axion generically lead to 1/4 BPS rather than 1/2 BPS solutions. Although we have not investigated the possibility of turning on a three-form, their non-trivial presence would almost certainly also lead to further reduction

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of supersymmetry from 1/2 BPS to 1/4 BPS. Thus we believe that the space of 1/2 BPS configurations will not be enlarged by turning on any additional fields.

However there is still the second observation, which is that the construction of [1] using the invariant tensor approach of [6–10] only follows the case where the Killing vector $K^\mu = \bar{\epsilon}\gamma^\mu\epsilon$ formed out of the Killing spinor ϵ is timelike. Here we recall that the classification and construction of such geometries generally fall into two classes, depending on whether the Killing vector is timelike or null. In the case of bubbling AdS geometries [1], the Killing vector K^μ is conjugate to the AdS energy (corresponding to conformal dimension Δ), so on physical grounds it ought to be timelike. However, as far as constructing all 1/2 BPS solutions is concerned, one ought to investigate the null case as well for completeness. In particular, we note that there are static solutions such as the magnetic string in five dimensions which fall in the null Killing vector category [8].

In this Letter we complete the search for all 1/2 BPS solutions of IIB supergravity preserving $SO(4) \times SO(4)$ isometry by directly analyzing the null Killing vector case of the invariant tensor construction. Note that this was also considered in [11]. The result is perhaps not so surprising: the only solutions falling in this case are the IIB pp-waves with appropriate isometry. We note that the particular case of the maximally supersymmetric pp-wave [12] is also obtained in the timelike case when the phase space configuration is identified with the half filled plane. Although our initial expectation, based on the analysis of theories with eight supercharges [6,8], was that the null case could capture a larger set of either pp-wave or plane-fronted wave solutions, we found that this in fact does not happen for IIB solutions with $SO(4) \times SO(4)$ isometry. Nevertheless, our analysis essentially completes the proof that all 1/2 BPS solutions of IIB supergravity with the given isometry are of the bubbling AdS form [1] (at least up to the remote possibility of turning on three-form flux without breaking any additional supersymmetry).

2. Supersymmetry analysis

As shown in [5], IIB supergravity admits a consistent bosonic $S^3 \times S^3$ breathing mode reduction of the form

$$ds_{10}^2 = g_{\mu\nu}(x) dx^\mu dx^\nu + e^{H(x)} (e^{G(x)} d\Omega_3^2 + e^{-G(x)} d\tilde{\Omega}_3^2),$$

$$F_{(5)} = F_{(2)} \wedge \omega_3 + \tilde{F}_{(2)} \wedge \tilde{\omega}_3, \quad (2.1)$$

which only retains the relevant fields, namely the metric and self-dual five-form. The resulting four-dimensional Lagrangian is given by

$$e^{-1}\mathcal{L} = e^{3H} \left[R + \frac{15}{2} \partial H^2 - \frac{3}{2} \partial G^2 - \frac{1}{4} e^{-3(H+G)} F_{\mu\nu}^2 + 12e^{-H} \cosh G \right]. \quad (2.2)$$

Although this system is not a consistent supergravity in four dimensions, it nevertheless has associated with it the reduction

of the IIB supersymmetry variations

$$\begin{aligned} \delta\psi_\mu &= \left[\nabla_\mu + \frac{i}{16} e^{-\frac{3}{2}(H+G)} F_{\nu\lambda} \gamma^{\nu\lambda} \gamma_\mu \right] \epsilon, \\ \delta\chi_H &= \left[\gamma^\mu \partial_\mu H + e^{-\frac{1}{2}H} (\eta e^{-\frac{1}{2}G} - i\tilde{\eta} \gamma_5 e^{\frac{1}{2}G}) \right] \epsilon, \\ \delta\chi_G &= \left[\gamma^\mu \partial_\mu G - \frac{i}{4} e^{-\frac{3}{2}(H+G)} F_{\mu\nu} \gamma^{\mu\nu} \right. \\ &\quad \left. + e^{-\frac{1}{2}H} (\eta e^{-\frac{1}{2}G} + i\tilde{\eta} \gamma_5 e^{\frac{1}{2}G}) \right] \epsilon, \end{aligned} \quad (2.3)$$

where the four-dimensional spinors are Dirac and $\eta = \pm 1$, $\tilde{\eta} = \pm 1$ are independent signs related to the Killing spinor orientations on the three-spheres.

The usefulness of this consistent reduction approach is that any solution to the four-dimensional system (2.2) may be lifted to obtain a IIB solution with the isometry of $S^3 \times S^3$. If, in addition, the solution satisfies the Killing spinor conditions deduced from (2.3), then the uplifted background is guaranteed to be supersymmetric. Working with this system is of course equivalent to working in the original framework of [1].

The supersymmetry analysis [1,5] proceeds by defining a complete set of Dirac bilinears

$$\begin{aligned} f_1 &= \bar{\epsilon}\gamma^5\epsilon, & f_2 &= i\bar{\epsilon}\epsilon, \\ K^\mu &= \bar{\epsilon}\gamma^\mu\epsilon, & L^\mu &= \bar{\epsilon}\gamma^\mu\gamma^5\epsilon, \\ Y_{\mu\nu} &= i\bar{\epsilon}\gamma_{\mu\nu}\gamma^5\epsilon, \end{aligned} \quad (2.4)$$

where ϵ is assumed to be a Killing spinor. By Fierz rearrangement, we may obtain the algebraic identities

$$L^2 = -K^2 = f_1^2 + f_2^2, \quad K \cdot L = 0. \quad (2.5)$$

As K^μ turns out to be a Killing vector with non-positive norm, solutions fall into two classes, depending on whether K^μ is timelike or null. The timelike case was thoroughly investigated in [1]. Hence we only concern ourselves with the null case in the present analysis.

From (2.5), it is clear that the null Killing vector case ($K^2 = 0$) corresponds to taking $f_1 = f_2 = 0$. Although it is clear from above that the vector L^μ has zero norm, we can further use the $\delta\chi_H = 0$ ‘differential’ identities to deduce that it is in fact vanishing, i.e. $L_\mu = 0$. In this case, the complete set of differential identities given in Appendix C of [5] collapse into the equivalent set

$$\begin{aligned} \eta K_\mu &= *Y_\mu{}^\nu \partial_\nu w_1, & *Y_\mu{}^\nu \partial_\nu w_2 &= 0, \\ 2K_{[\mu} \partial_{\nu]} w_1 &= \eta *Y_{\mu\nu}, \\ \tilde{\eta} K_\mu &= Y_\mu{}^\nu \partial_\nu w_2, & Y_\mu{}^\nu \partial_\nu w_1 &= 0, & 2K_{[\mu} \partial_{\nu]} w_2 &= \tilde{\eta} Y_{\mu\nu}, \\ K^\mu \partial_\mu w_1 &= K^\mu \partial_\mu w_2 = 0, & K^\mu F_{\mu\nu} &= K^\mu *F_{\mu\nu} = 0, \\ F_{\mu\nu} Y^{\mu\nu} &= F_{\mu\nu} *Y^{\mu\nu} = F_\mu{}^\lambda Y_{\nu\lambda} = 0, \\ \nabla_\mu K_\nu &= \nabla_\mu Y_{\nu\lambda} = 0. \end{aligned} \quad (2.6)$$

Here we have defined the scalar combinations

$$w_1 \equiv e^{\frac{1}{2}(H+G)}, \quad w_2 \equiv e^{\frac{1}{2}(H-G)}. \quad (2.7)$$

2.1. Specialization of the metric

We now note from (2.6) that K^μ is both null and covariantly constant. This allows us to introduce null coordinates (u, v) such that

$$K^\mu \partial_\mu = \frac{\partial}{\partial v}, \quad K_\mu dx^\mu = du. \quad (2.8)$$

The resulting metric may be written in the form of a pp-wave

$$ds^2 = 2 du dv + \mathcal{F} du^2 + \Omega^2 (dy_1^2 + dy_2^2) = e^+ e^- + e^i e^i, \quad (2.9)$$

where

$$e^+ = du, \quad e^- = dv + \frac{1}{2} \mathcal{F} du, \quad e^i = \Omega dy_i. \quad (2.10)$$

For simplicity, we have chosen the two-dimensional transverse metric to be conformally flat. Furthermore, both functions \mathcal{F} and ω are independent of v .

We may now use the identities $\tilde{\eta}Y = K \wedge dw_2$ and $\eta * Y = K \wedge dw_1$ from (2.6) along with (2.8) to demonstrate that the two-form Y takes the form

$$Y = (\tilde{\eta} \partial_i w_2) du \wedge dy_i = (\eta \epsilon_{ij} \partial_j w_1) du \wedge dy_i, \quad (2.11)$$

where we recall that the signs η and $\tilde{\eta}$ are of unit magnitude. We note that (2.11) implies the Cauchy–Riemann equations for (w_1, w_2)

$$\partial_i w_2 = \eta \tilde{\eta} \epsilon_{ij} \partial_j w_1 \quad (2.12)$$

(which further ensures that they are harmonic in the y_1 – y_2 plane).

Using $\eta K_\mu = *Y_\mu{}^v \partial_v w_1$ and $\tilde{\eta} K_\mu = Y_\mu{}^v \partial_v w_2$ from (2.6) now allows us to obtain the conformal factor

$$\Omega^2 = \partial_i w_1 \partial_i w_1 = \partial_i w_2 \partial_i w_2. \quad (2.13)$$

The Cauchy–Riemann equations along with the solution for the conformal factor allow us to make the observation

$$\Omega^2 (dy_1^2 + dy_2^2) = (dw_1 - \partial_u w_1 du)^2 + (dw_2 - \partial_u w_2 du)^2. \quad (2.14)$$

As a result, it becomes natural to perform a $(u$ -dependent) coordinate transformation from y_i to x_i according to

$$x_1 = w_1(u, y_1, y_2), \quad x_2 = -\eta \tilde{\eta} w_2(u, y_1, y_2), \quad (2.15)$$

where the sign factors in the x_2 transformation are chosen for later convenience. The result of this transformation is to essentially make use of (w_1, w_2) as coordinates on the base. This manipulation parallels the observation presented in [8] that a preferred set of coordinates may be chosen on the base for the null case.

The above coordinate transformation allows us to further specialize the metric (2.9) to the particularly simple form

$$ds^2 = 2 du dv + \mathcal{F} du^2 + (dx_1 - a_1 du)^2 + (dx_2 - a_2 du)^2. \quad (2.16)$$

Corresponding to this metric, we have

$$w_1 = x_1, \quad w_2 = -\eta \tilde{\eta} x_2, \quad Y = -\eta du \wedge dx^2. \quad (2.17)$$

A note on the choice of signs is now in order. We recall from (2.7) that both w_1 and w_2 are non-negative. As a result, (x_1, x_2) are restricted to a single quadrant of the plane. We may choose both x_1 and x_2 to be positive, in which case the signs must obey the condition $\eta = -\tilde{\eta} = \pm 1$. As in [1], this two-fold degeneracy of signs is important in ensuring that the solution is overall 1/2 BPS and not 1/4 BPS.

Before obtaining the two-form field strength F , we first constrain the vector $\vec{a}(u, x_1, x_2)$ on the base. Perhaps the most straightforward way to do so is to enforce the covariant constancy of Y indicated in (2.6). In particular, $\nabla_u Y_{u1} = 0$ demonstrates that \vec{a} is curl-free

$$\epsilon_{ij} \partial_i a_j = 0. \quad (2.18)$$

As a result, \vec{a} may be written as a divergence, $\vec{a} = \vec{\nabla} \varphi$. In this case, \vec{a} may be removed by a coordinate transformation

$$v \rightarrow v + \varphi(u, x_1, x_2), \quad (2.19)$$

along with a redefinition

$$\mathcal{F} \rightarrow \mathcal{F} - 2\partial_u \varphi - (\partial_i \varphi)^2. \quad (2.20)$$

The metric is now of a conventional pp-wave form

$$ds^2 = 2 du dv + \mathcal{F} du^2 + dx_1^2 + dx_2^2. \quad (2.21)$$

3. Completing the solution

To complete the solution, we now turn to the two-form field strength F . From $K^\mu F_{\mu\nu} = K^\mu * F_{\mu\nu} = 0$ given in (2.6), we see that F only has components along $du \wedge dx^i$. However, using $F_\mu{}^\lambda Y_{\nu\lambda} = 0$ along with the explicit form of Y in (2.17), we see that the $du \wedge dx^2$ component must vanish. As a result, F may be written as

$$F = F_{u1}(u, x_1, x_2) du \wedge dx^1. \quad (3.1)$$

In writing the field strength (3.1) and metric (2.21), we have now exhausted the content of the null case differential identities (2.6). To guarantee a valid solution, however, we must also impose the Bianchi identity and equation of motion on F ,

$$dF = 0, \quad d(w_1^{-3} w_2^3 * F) = 0. \quad (3.2)$$

Using (3.1), these are equivalent to

$$\partial_2 F_{u1} = 0, \quad \partial_1 ((x_2/x_1)^3 F_{u1}) = 0. \quad (3.3)$$

As a result, F is given by

$$F = x_1^3 f(u) du \wedge dx^1. \quad (3.4)$$

For completeness, we note here that the Killing spinor ϵ is of the form

$$\epsilon(u) = \exp\left(-\frac{i}{4} \int^u f(u) du\right) \epsilon_0, \quad \gamma^+ \epsilon_0 = 0, \quad \gamma^1 \epsilon_0 = \epsilon_0, \quad (3.5)$$

where ϵ_0 is a constant spinor.

Finally, as expected for a pp-wave solution, the uu component of the Einstein equation is undetermined by supersymmetry, and remains to be imposed. In the present case, this equation has the form

$$(x_1 x_2)^{-3} \partial_i ((x_1 x_2)^3 \partial_i \mathcal{F}) = -f(u)^2, \quad (3.6)$$

and admits a solution

$$\mathcal{F} = -\frac{1}{16} (x_1^2 + x_2^2) f(u)^2 + \mathcal{F}_0(u, x_1, x_2), \quad (3.7)$$

where \mathcal{F}_0 is any solution to the Laplace equation

$$\partial_i ((x_1 x_2)^3 \partial_i \mathcal{F}_0) = 0. \quad (3.8)$$

The solution is now complete, and is specified by two functions: an arbitrary function $f(u)$, and a harmonic (in x_1, x_2) function $\mathcal{F}_0(u, x_1, x_2)$. Although we have worked with an effective four-dimensional description, (2.2), it is instructive to write down the lifted null solution

$$ds^2 = 2 du dv + \mathcal{F} du^2 + [dx_1^2 + x_1^2 d\Omega_3^2] + [dx_2^2 + x_2^2 d\tilde{\Omega}_3^2], \\ F_5 = x_1^3 f(u) du \wedge dx_1 \wedge \omega_3 + x_2^3 f(u) du \wedge dx_2 \wedge \tilde{\omega}_3, \quad (3.9)$$

where \mathcal{F} is given by (3.7) and (3.8). It is now clear that we have precisely reproduced the IIB pp-wave [12] under the constraint of $S^3 \times S^3$ invariance. The generic pp-wave of this form preserves 16 of the original 32 supersymmetries, although the maximally supersymmetric case may be obtained by taking $f(u) = \text{constant}$ and $\mathcal{F}_0 = 0$ [12].

We note that the generic IIB pp-wave does not fall into the LLM bubbling picture of [1], although the latter does allow for 1/2 BPS excitations of plane wave geometries. The most direct way to see that the pp-wave excitations are distinct is to perform the change of variables

$$u = t, \quad v = \hat{x}_1, \quad y = x_1 x_2, \quad \hat{x}_2 = \frac{1}{2} x_1^2 - x_2^2, \quad (3.10)$$

on the ten-dimensional metric (3.9). The resulting IIB geometry then has the LLM form

$$ds^2 = -h^{-2} (dt - h^2 d\hat{x}_1)^2 + \hat{\mathcal{F}} dt^2 + h^2 (d\hat{x}_1^2 + d\hat{x}_2^2 + dy^2) \\ + y [e^G d\Omega_3^2 + e^{-G} d\tilde{\Omega}_3^2], \quad (3.11)$$

where

$$h^{-2} = 2y \cosh G, \quad e^G = \hat{x}_2/y + \sqrt{(\hat{x}_2/y)^2 + 1}, \quad (3.12)$$

and

$$\hat{\mathcal{F}} = \left(1 - \frac{1}{16} f(t)^2\right) h^{-2} + \mathcal{F}_0. \quad (3.13)$$

It is now evident that only when $\hat{\mathcal{F}} = 0$ (corresponding to the maximally supersymmetric plane wave) does the above solution admit a strict LLM interpretation [11]. This furthermore demonstrates that the bubbling excitations above the plane wave background (i.e. the half filled \hat{x}_1 – \hat{x}_2 plane) fall only into the timelike Killing vector category. This is the case, even for the constant energy density excitations, which admit the Killing vector $\partial/\partial \hat{x}_1$.

4. Conclusions

We have thus demonstrated that all 1/2 BPS bubbling AdS solutions (or, more precisely, IIB solutions with $SO(4) \times SO(4)$ isometry) in the null Killing vector case are of the form of IIB pp-waves. This null case analysis complements the time-like case investigated in [1], and essentially completes the study of 1/2 BPS solutions of IIB supergravity with $SO(4) \times SO(4)$ isometry, at least at the local level [4]. Except for the remote possibility that three-form flux could be turned on without breaking any further supersymmetry, we have essentially completed the proof that *all* 1/2 BPS solutions of IIB supergravity with the given isometry are either of bubbling AdS [1] or pp-wave [12] form.

We note that this analysis of the null Killing vector case of IIB solutions with $SO(4) \times SO(4)$ isometry can also be extended to the bubbling AdS_3 case of minimal six-dimensional supergravity with T^2 isometry [13,14]. However, this $\mathcal{N} = (1, 0)$ theory is very much simpler than IIB supergravity, and in fact all its supersymmetric vacua have already been classified [9]. (This is in fact how the bubbling geometries of [13] were obtained.) Thus there is little to be gained from working out the null analysis, except perhaps a more explicit means of writing out solutions in a preferred coordinate frame.

Although this exhausts the local classification of 1/2 BPS configurations with $SO(4) \times SO(4)$ isometry, the full classification remains open once the isometry constraint is relaxed. As mentioned above, the general parallel D3-brane supergravity solution is an example of a perfectly regular 1/2 BPS solution which falls outside of the present classification. It would certainly be worthwhile, as well as instructive, to obtain a complete construction of all regular 1/2 BPS solutions of IIB supergravity. This construction appears highly non-trivial, as without any underlying isometry assumption, we have all ten dimensions to consider. In contrast, the use of $SO(4) \times SO(4)$ isometry reduced the problem to one of an effective four-dimensional system, and the application of the invariant tensor method for constructing such lower dimensional backgrounds is presently very well understood. (Even though this four-dimensional truncation is not a consistent supergravity, all that is needed is a manageable set of Killing spinor conditions, which this system admits, in order to proceed with the construction.)

While we were initially motivated to seek out novel bubbling AdS solutions with null isometries, we have ended up focusing on the classification and potential construction of new 1/2 BPS solutions regardless of the existence of any gauge theory duals. This classification program has been an extremely rich one, and there are of course a much larger variety of possibilities than just 1/2 BPS structures [15,16]. In fact, one of our ultimate aims would be to relax the $SO(4) \times SO(4)$ isometry, and thereby examine backgrounds with fewer supersymmetries. We are especially interested in 1/8 BPS configurations, as they would encompass many important systems such as $\mathcal{N} = 1$ flux compactifications and models of black hole microstates, as well as reduced supersymmetry sectors of the gauge/gravity correspondence.

Acknowledgements

This work was initiated during a visit to the Khuri lab at the Rockefeller University. J.T.L. wishes to acknowledge the hospitality of the KITP where this work was completed. This work was supported in part by the US Department of Energy under grant DE-FG02-95ER40899 and by the National Science Foundation under grants PHY99-07949 and PHY06-01213.

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